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AN ACOUSTIC METHOD OF DETERMINATION OF THE COEFFICIENT OF PERMEABILITY AND THE PORE VELOCITY OF SOUND IN SNOW

M. K. Zhekamukhov and L. M. Malkandueva

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Using an experimental setup that allows one to distinguish resonance frequencies of vibrations of the studied specimen of snow, we showed the possibility of determining the coefficient of permeability of snow and the velocity of sound in the air confined in its pores.

The practice of prediction of snowslides requires on-line data of the characteristics of strength of the snow cover. Acoustic methods based on measurements of the velocity of sound in snow by recording the time of travel of an ultrasonic pulse through a snow specimen of known size are widely used for obtaining such data. However, this method with one receiver does not allow separate recording of the parameters of waves propagating through the solid skeleton of snow and the air filling the pores in snow, since these parameters can be of the same order.

In [1], an original experimental setup is suggested (Fig. 1); this setup allows one, by exciting the systems of standing waves in the studied snow specimen, to separately distinguish resonance frequencies for sound vibrations propagating through the solid skeleton of the snow specimen and the air filling the pores in snow.

In the present paper, on the basis of theoretical calculations we show that by using the above-mentioned setup we can also determine the coefficient of permeability of snow and the velocity of sound in the air confined in snow pores (pore velocity of sound). The setup includes a thin-walled pipe with a smooth inner surface into which the studied specimen of snow is placed. In small deformation of snow there is no lateral expansion in this tube.

The schematic (Fig. 1) shows the generator of sound frequencies 1, which is a source of alternating voltage and serves as a power source of the electromagnetic exciter 2, piston membrane 3, acoustic pipe 4 with the snow specimen 5, receivers 6 and 7, pre-amplifiers 8 and 9, double-beam oscilloscope 10, and frequency meter 11.

The piston membrane vibrates under the action of the generator, and these vibrations are transferred to the snow specimen adjacent to it. The receivers take elastic vibrations and transform them to electric oscillations, which, after amplification, are sent to the oscilloscope. Two receivers are used: one of them is a miniature electrodynamic microphone and the other — a piezoceramic sensor with a recording stylus.

In smooth variation of the frequency of exciting acoustic vibrations the instant sets in when this frequency coincides initially with the first, main, frequency, then with the second, and so on, frequencies of natural vibrations of the studied snow specimen. Resonance frequencies are determined by maxima of the deviation of spots on the oscilloscope display.

In sounding the specimen, the membrane is in contact with the snow. A small gap (of about 0.1 mm) is left between the body of the microphone and the snow surface such that vibrations of the snow skeleton were not transferred to the microphone (the piezosensor does not react to air vibrations). The stylus of the piezosensor touches the surface of the snow specimen and is frozen to it.

Experimental studies conducted in [1] showed that the considered setup allows one to clearly record the first three to four resonance frequencies of elastic vibrations propagating through both the solid skeleton of snow and the air filling the snow pores.

Resonance frequencies of vibrations $\nu_{n\mathrm{r}}$ that coincide with natural frequencies of vibrations of the solid skeleton of the snow specimen are related to the longitudinal velocity of sound c_l in the solid skeleton of snow as follows:

Kabardino-Balkar State University, 173 Chernyshevskii Str., Nal'chik, 360000, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 77, No. 6, pp. 131–140, November–December, 2004. Original article submitted August 6, 2003.

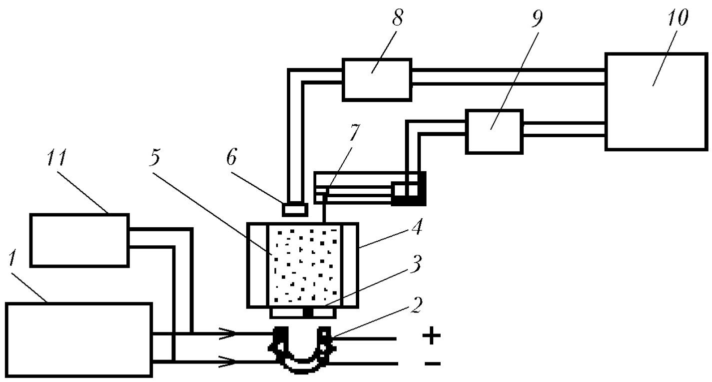

Fig. 1. Schematic of the setup for measuring the parameters of elastic waves in snow specimens by the method of a resonance pipe.

$$\nu_{n\mathrm{r}} = \frac{nc_\ell}{2h}, \quad n = 1, 2, 3, \dots . \tag{1}$$

On the other hand, the velocity of propagation of sound through the solid skeleton of snow is related to the Young modulus E as

$$c_\ell = \sqrt{\frac{(1-\sigma)\, E}{\rho\,(1+\sigma)\,(1-2\sigma)}} . \tag{2}$$

Thus, formulas (1) and (2) allow one to determine the Young modulus E by resonance frequencies if the snow density and the Poisson coefficient are known.

Formula (1), however, is inapplicable to air inclusions of snow, since the latter, as was shown in [2], form a dispersive medium where each monochromatic wave with frequency ω has its own velocity of sound $v_\mathrm{g} = v_\mathrm{g}(\omega)$. As will be shown below, resonance frequencies of vibrations of the air confined in the snow pores are directly related to the coefficient of permeability of snow. Our task is to determine these relations.

Forced Longitudinal Vibrations of Snow between Two Parallel Planes. As a theoretical model of forced vibrations of the snow specimen placed in a steel pipe in the above-described setup we consider forced vibrations of a snow layer of thickness h between two infinite parallel planes (Fig. 2). In this snow, as well as in the snow in the steel pipe, there is no lateral expansion.

We assume that starting from the time instant $t = 0$, a unit surface area of the lower base of the snow layer is affected by the periodic force

$$F\,(t) = F_0 \sin \omega t . \tag{3}$$

The system of equations describing longitudinal vibrations of snow in displacements is written as [2]

$$\frac{\partial^2 u_1}{\partial t^2} = c_\ell^2 \frac{\partial^2 u_1}{\partial x^2} + \varepsilon' c_2^2 \frac{\partial^2 u_2}{\partial x^2} + \varepsilon \omega^* \left(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \right), \quad 0 < x < h ; \tag{4}$$

$$\frac{\partial^2 u_2}{\partial t^2} = c_2^2 \frac{\partial^2 u_2}{\partial x^2} + \left(\frac{1}{f_0} - 1 \right) \frac{\partial^2 u_1}{\partial t^2} + \omega^* \left(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \right), \quad 0 < x < h . \tag{5}$$

In Eq. (4), the terms involving the factors ε and ε′ are small compared to the other terms and they can be discarded. Physically this means that the resistance that the solid skeleton of snow experiences from the side of the air

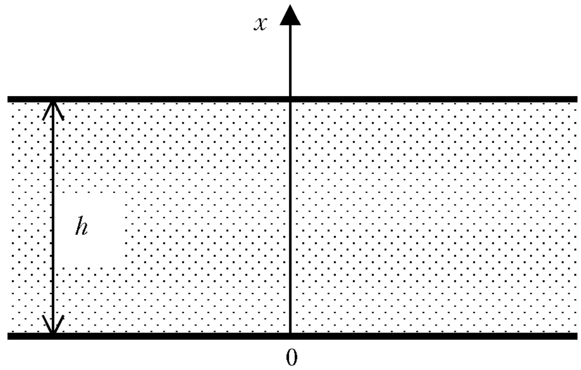

Fig. 2. Snow specimen placed between the parallel planes.

filling the pores between the ice crystals in snow is negligibly small; consequently the presence of air in the pores of snow virtually does not affect the elastic vibrations of the snow skeleton.

Thus, for the solid skeleton of snow we have

$$\frac{\partial^2 u_1}{\partial t^2} = c_\ell^2 \frac{\partial^2 u_1}{\partial x^2} , \quad 0 < x < h . \tag{6}$$

The solution of Eq. (6) must satisfy the initial and boundary conditions

$$u_1 (x, 0) = u_{1t} (x, 0) = 0 , \tag{7}$$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = A \sin \omega t , \quad \left.\frac{\partial u_1}{\partial t}\right|_{x=h} = 0 , \tag{8}$$

where $A = F_0 / E$.

We seek the solution of problem (6)–(8) in the form of the sum of two functions:

$$u_1 (x, t) = Ax \left(1 - \frac{x}{2h}\right) \sin \omega t + V (x, t) . \tag{9}$$

Here $V(x, t)$ is a new unknown function that satisfies the nonhomogeneous wave equation

$$\frac{\partial^2 V}{\partial t^2} = c_\ell^2 \frac{\partial^2 V}{\partial x^2} + A \left[\omega^2 x \left(1 - \frac{x}{2h}\right) - \frac{c_\ell^2}{h} \right] \sin \omega t , \tag{10}$$

the homogeneous boundary equations

$$\frac{\partial V (0, t)}{\partial x} = \frac{\partial V (h, t)}{\partial x} = 0 , \tag{11}$$

and the initial conditions

$$V (x, 0) = 0 , \quad V_t (x, 0) = - A \omega x \left(1 - \frac{x}{2h}\right) . \tag{12}$$

The solution of problem (10)–(12) can be written in the form (see, e.g., [3])

$$V(x,t)=\sum_{n=1}^{\infty} B_n \cos\omega_n t - \cos\frac{\pi n x}{h} - \frac{2A\omega^2}{h}\times$$

$$\times\sum_{n=1}^{\infty}\frac{1}{\omega_n}\left[\int_0^h\left(\frac{c_\ell^2}{h\omega^2}-\xi+\frac{\xi^2}{2h}\right)\cos\frac{\pi n\xi}{h}\,d\xi\int_0^t \sin\omega\tau\sin\omega_n(t-\tau)\,d\tau\right]\cos\frac{\pi n x}{h}, \tag{13}$$

where $\omega_n = \pi n c_l/h$ and the coefficients B_n are determined as follows:

$$B_n = -\frac{2A\omega}{h\omega_n}\int_0^1 \xi\left(1-\frac{\xi}{2h}\right)\cos\frac{\pi n\xi}{h}\,d\xi = \frac{2A\omega h}{\pi^2 n^2\omega_n}. \tag{14}$$

The integrals in (13) are

$$\int_0^h\left(\frac{c_\ell^2}{h\omega^2}-\xi+\frac{\xi^2}{2h}\right)\cos\frac{\pi n\xi}{h}\,d\xi = \frac{h^2}{\pi^2 n^2}, \tag{15}$$

$$\int_0^t \sin\omega\tau\sin\omega_n(t-\tau)\,d\tau = \frac{\omega_n\sin\omega t-\omega\sin\omega_n t}{\omega^2-\omega_n^2}. \tag{16}$$

Substituting expressions (14)–(16) into (13) and allowing for equality (9), we obtain

$$u_1(x,t) = Ax\left(1-\frac{x}{2h}\right)\sin\omega t + \frac{2Ah\omega}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2\omega_n}\cos\omega_n t\cos\frac{\pi n x}{h} -$$

$$-\frac{2Ah\omega^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2\omega_n}\frac{\omega_n\sin\omega t-\omega\sin\omega_n t}{\omega^2-\omega_n^2}\cos\frac{\pi n x}{h}, \tag{17}$$

where $\omega \neq \omega_n$.

It is seen from this solution that resistance frequencies for the snow specimen are

$$\omega_{nr} = \frac{\pi n c_\ell}{h}, \quad n = 1, 2, 3, \ldots .$$

In what follows, we consider forced vibrations of the air confined in pores of the studied snow specimen. In these vibrations, viscous forces play an important role; therefore, all terms must be retained in Eq. (5). In this case, ice crystals executing forced vibrations described by formula (17) produce additional bulk forces, which affect the air filling the free space between the ice crystals. Thus, solution of (5) must satisfy the same initial and boundary conditions as $u_1(x, t)$:

$$u_2(x,0) = u_{2t}(x,0) = 0, \tag{18}$$

$$\left.\frac{\partial u_2}{\partial x}\right|_{x=0} = A\sin\omega t, \quad \left.\frac{\partial u_2}{\partial x}\right|_{x=h} = 0. \tag{19}$$

We seek it in the form

$$u_2(x,t) = Ax\left(1-\frac{x}{2h}\right)\sin\omega t + W(x,t)\,.$$

Here $W(x, t)$ is the solution of the inhomogeneous wave equation

$$\frac{\partial^2 W}{\partial t^2} = c_2^2\frac{\partial^2 W}{\partial x^2} + A\omega^2 x\left(1-\frac{x}{2h}\right)\sin\omega t - \frac{Ac_2^2}{h}\sin\omega t + \left(\frac{1}{f_0}-1\right)\frac{\partial^2 u_1}{\partial t^2} +$$

$$+\,\omega^*\left(\frac{\partial u_1}{\partial t}-\frac{\partial W}{\partial t}\right) - A\omega\omega^* x\left(1-\frac{x}{2h}\right)\cos\omega t\,, \tag{20}$$

which satisfies the conditions

$$\frac{\partial W(0,t)}{\partial x} = \frac{\partial W(h,t)}{\partial x} = 0\,, \tag{21}$$

$$W(x,0) = 0\,,\quad \frac{\partial W(x,0)}{\partial t} = -A\omega x\left(1-\frac{x}{2h}\right). \tag{22}$$

Substituting the values of $\frac{\partial u_1}{\partial t}$ and $\frac{\partial^2 u_1}{\partial t^2}$ obtained from (17) into (20), we have

$$\frac{\partial^2 W}{\partial t^2} = c_2^2\frac{\partial^2 W}{\partial x^2} - \omega^*\frac{\partial W}{\partial t} + F_1(x,t)\,, \tag{23}$$

where

$$F_1(x,t) = \left[\omega^2 x\left(1-\frac{x}{2h}\right) - \frac{c_2^2}{h} - \left(\frac{1}{f_0}-1\right)\omega^2 x\left(1-\frac{x}{2h}\right) +\right.$$

$$\left. + \left(\frac{1}{f_0}-1\right)\frac{2h\omega^4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{\omega^2-\omega_n^2}\cos\frac{\pi n x}{h}\right]A\sin\omega t + A\omega\omega^* x\left(1-\frac{x}{2h}\right) -$$

$$-\frac{2Ah\omega^*\omega^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{1}{\omega^2-\omega_n^2}\cos\frac{\pi n x}{h}\cos\omega t - \frac{2Ah\omega^*\omega}{\pi^2}\sum_{n=1}^{\infty}\cos\frac{\pi n x}{h}\sin\omega_n t -$$

$$-\frac{2Ah\omega}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left(\frac{1}{f_0}-1\right)\omega_n\left(1+\frac{\omega^2}{\omega^2-\omega_n^2}\right) - \frac{\omega^2\omega^*}{\omega^2-\omega_n^2}\right]\cos\omega_n t\cos\frac{\pi n x}{h}\,.$$

We also seek the solution of Eq. (23) in the form of the sum of two functions:

$$W(x,t) = W_1(x,t) + W_2(x,t)\,,$$

where $W_1(x, t)$ is the solution of the wave equation

$$\frac{\partial^2 W_1}{\partial t^2} = c_2^2 \frac{\partial^2 W_1}{\partial x^2} - \omega^* \frac{\partial W_1}{\partial t} ,$$

satisfying conditions (21) and (22). It involves the exponential factor exp $(-\omega^* t)$ and attenuates with time; therefore, it is of no interest to us.

The function $W_2(x, t)$ is the solution of the inhomogeneous wave equation

$$\frac{\partial^2 W_2}{\partial t^2} = c_2^2 \frac{\partial^2 W_2}{\partial x^2} - \omega^* \frac{\partial W_2}{\partial t} + F_1(x, t) , \tag{24}$$

which satisfies the homogeneous initial and boundary conditions

$$\frac{\partial W_2(0, t)}{\partial x} = \frac{\partial W_2(h, t)}{\partial x} = 0 , \tag{25}$$

$$W_2(x, 0) = W_{2t}(x, 0) = 0 . \tag{26}$$

The functions

$$X_m(x) = \cos \frac{\pi m x}{h} , \quad m = 1, 2, 3, \ldots . \tag{27}$$

are the eigenfunctions of problem (24)–(26).

We seek the solution of Eq. (24) in the form of a series in terms of eigenfunctions (27):

$$W_2(x, t) = \sum_{m=1}^{\infty} C_m(t) \cos \frac{\pi m x}{h} , \tag{28}$$

where $C_m(t)$ are the unknown amplitudes of vibrations, which depend on time t and satisfy the initial conditions $C_m(0)$ $= \frac{dC_m(0)}{dt} = 0$.

Expansion of the function $F_1(x, t)$ into a series in terms of eigenfunctions yields

$$F_1(x, t) = \sum_{m=1}^{\infty} f_m(t) \cos \frac{\pi m x}{h} , \tag{29}$$

where

$$f_m(t) = \frac{2}{h} \int_0^h F_1(\xi, t) \cos \frac{\pi m \xi}{h} \, d\xi .$$

Substituting expressions (28) and (29) into Eq. (24), we obtain linear inhomogeneous differential equations of the second order relative to unknown functions $C_m(t)$:

$$\frac{d^2 C_m}{dt^2} + \omega^* \frac{dC_m}{dt} + \omega_m'^2 C_m = f_m(t) , \tag{30}$$

here

$$\omega'_m = \frac{\pi m c_2}{h}\,, \quad m = 1, 2, 3, \ldots \tag{31}$$

are the natural frequencies of vibrations of the air layer of thickness h in the absence of the solid snow skeleton.

The solutions of Eq. (30) that satisfy the zero initial conditions have the form

$$C_m(t) = \frac{1}{\overline{\omega}_m} \exp\left(-\frac{\omega^* t}{2}\right) \int_0^t F_1(\tau) \exp\left(\frac{\omega^* \tau}{2}\right) \sin \overline{\omega}_m (1 - \tau)\, d\tau\,, \tag{32}$$

where $\overline{\omega}_m = \sqrt{\omega_m'^2 - \frac{\omega^{*2}}{4}}$.

Vibration of the air confined in the snow pores is possible only when $\overline{\omega} > 0$, i.e., if $\frac{\omega^*}{\omega'_m} < 2$, whence it follows that

$$\frac{k}{h} > \frac{\nu_2}{2\pi m c_2}\,, \quad m = 1, 2, 3, \ldots\,. \tag{33}$$

Thus, for standing waves to appear in the air confined in the pores of the considered specimen of snow, it is necessary that the coefficient of permeability of snow be rather large or, which is the same, that the snow density be smaller than some critical quantity.

In substituting the value of $F_1(\tau)$ into (32) a number of integrals appear, which have the form

$$J_1 = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \sin \omega\tau \sin \overline{\omega}_m (t - \tau)\, d\tau\,, \quad J_2 = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \cos \omega\tau \sin \overline{\omega}_m (t - \tau)\, d\tau\,,$$

$$J_3 = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \sin \omega_n\tau \sin \overline{\omega}_m (t - \tau)\, d\tau\,, \quad J_4 = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \cos \omega_n\tau \sin \overline{\omega}_m (t - \tau)\, d\tau\,.$$

The integrals J_3 and J_4 are found from J_1 and J_2 by substitution of ω by ω_n; therefore, it suffices to consider the first two.

We present J_1 in the form

$$J_1 = J_1' \sin \overline{\omega}_m t - J_1'' \cos \overline{\omega}_m t\,,$$

where

$$J_1' = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \sin \omega\tau \cos \overline{\omega}_m \tau d\tau = \frac{1}{2} \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) [\sin (\omega + \overline{\omega}_m)\, \tau + \sin (\omega - \overline{\omega}_m)\, \tau]\, d\tau\,;$$

$$J_1'' = \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) \sin \omega\tau \sin \overline{\omega}_m \tau d\tau = \frac{1}{2} \int_0^t \exp\left(\frac{\omega^* \tau}{2}\right) [\cos (\omega + \overline{\omega}_m)\, \tau - \cos (\omega + \overline{\omega}_m)\, \tau]\, d\tau\,.$$

The obtained integrals are calculated by the table formulas [4]

$$\int \exp(ax)\sin bx\, dx = \frac{\exp(ax)}{a^2+b^2}(a\sin bx - b\sin bx)\,,$$

$$\int \exp(ax)\cos bx\, dx = \frac{\exp(ax)}{a^2+b^2}(a\cos bx + b\sin bx)\,.$$

Then

$$J_1 = \frac{1}{2}\exp\left(\frac{\omega^* t}{2}\right)\left[\frac{\frac{\omega^*}{2}\cos\omega t + (\omega+\bar{\omega}_m)\sin\omega t}{\frac{\omega^{*2}}{4}+(\omega+\bar{\omega}_m)^2} - \frac{\frac{\omega^*}{2}\cos\omega t + (\omega-\bar{\omega}_m)\sin\omega t}{\frac{\omega^{*2}}{4}+(\omega-\bar{\omega}_m)^2}\right] +$$

$$+\frac{1}{2}\left[\frac{(\omega-\bar{\omega}_m)}{\frac{\omega^{*2}}{4}+(\omega-\bar{\omega}_m)^2} - \frac{(\omega+\bar{\omega}_m)}{\frac{\omega^{*2}}{4}+(\omega+\bar{\omega}_m)^2}\right]\sin\bar{\omega}_m t +$$

$$+\frac{\omega^*}{4}\left[\frac{1}{\frac{\omega^{*2}}{4}+(\omega-\bar{\omega}_m)^2} - \frac{1}{\frac{\omega^{*2}}{4}+(\omega+\bar{\omega}_m)^2}\right]\cos\bar{\omega}_{mt}\,.$$

Introducing the notation

$$\Delta_m = \left[\frac{\omega^{*2}}{4}+(\omega-\bar{\omega}_m)^2\right]\left[\frac{\omega^{*2}}{4}+(\omega-\bar{\omega}_m)^2\right] = (\omega^2-\omega_m'^2)^2+\omega^2\omega^{*2}\,,$$

we can present this expression in the form

$$J_1 = \frac{\bar{\omega}_m}{\sqrt{\Delta_m}}\exp\left(\frac{\omega^* t}{2}\right)\sin(\omega t-\delta_m) + \frac{\omega}{\Delta_m}\left[\left(\frac{\omega^{*2}}{4}+\omega^2-\omega_m'^2\right)\sin\bar{\omega}_m t + \omega^*\bar{\omega}_m\cos\omega_m t\right], \tag{34}$$

where $\delta_m = \arctan\dfrac{\omega\omega^*}{\omega^2-\omega_m'^2}$. Substituting in the integral J_2 the difference $t-\tau$ by ξ, we obtain

$$J_2 = \exp\left(\frac{\omega^* t}{2}\right)\int_0^t \exp\left(-\frac{\omega^*\xi}{2}\right)\sin\bar{\omega}_m\xi\cos\omega(t-\xi)\,d\xi\,.$$

In this expression, the integral is calculated in the same way as the integral J_1:

$$J_2 = -\frac{\bar{\omega}_m}{\sqrt{\Delta_m}}\exp\left(\frac{\omega^* t}{2}\right)\sin(\omega t+\delta_m) - \frac{\omega}{\Delta_m}\left[\left(\omega^2-\omega_m'^2\right)\cos\bar{\omega}_m t + \bar{\omega}\omega^*\sin\omega t\right]. \tag{35}$$

Substituting in equalities (34) and (35) ω by ω_n, we obtain the values of the integrals J_3 and J_4. In this case, instead of Δ_m we have Δ_{mn} in the form

$$\Delta_{mn} = \left(\omega_n^2-\omega_m'^2\right)^2 + \omega_n^2\omega^{*2}\,.$$

Substitution of the values of the integrals J_1, J_2, J_3, and J_4 into (32) gives two groups of terms, one of which involves the exponential factor $\exp\left(-\frac{\omega^* t}{2}\right)$ (these terms vanish with time) and the second of which does not have this factor and describes the set-in forced vibrations of the air confined in the snow pores. This group, in turn, involves the terms with factors $\frac{1}{\sqrt{\Delta_m}}$ and $\frac{1}{\sqrt{\Delta_{mn}}}$, which can be presented as

$$\frac{1}{\Delta_m} = \frac{1}{\sqrt{(\omega^2 - \omega_m'^2)^2 + \omega^2 \omega^{*2}}} = \frac{1}{\omega_m'^2} \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_m'}\right)^2\right]^2 + \left(\frac{\omega}{\omega_m'}\right)^2 \left(\frac{\omega^*}{\omega_m'}\right)^2}},$$

$$\frac{1}{\Delta_{mn}} = \frac{1}{\sqrt{(\omega_n^2 - \omega_m'^2)^2 + \omega_n^2 \omega^{*2}}} = \frac{1}{\omega_m'^2} \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_n}{\omega_m'}\right)^2\right]^2 + \left(\frac{\omega_n}{\omega_m'}\right)^2 \left(\frac{\omega^*}{\omega_m'}\right)^2}}.$$

In the theory of vibrations, the coefficient

$$\lambda_m = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_m'}\right)^2\right]^2 + \left(\frac{\omega}{\omega_m'}\right)^2 \left(\frac{\omega^*}{\omega_m'}\right)^2}} \tag{36}$$

is called the dynamic-response factor [5]. It shows how many times the amplitudes of forced vibrations of the system in resonance exceed static displacement of the points of the system under the effect of the constant force that in magnitude is equal to the amplitude F_0 of the disturbing force.

Substituting in (36) ω by ω_n, we obtain the second dynamic-response factor λ_{mn}:

$$\lambda_{mn} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_n}{\omega_m'}\right)^2\right]^2 + \left(\frac{\omega_n}{\omega_m'}\right)^2 \left(\frac{\omega^*}{\omega_m'}\right)^2}}.$$

Introducing the notation $z_m = \frac{\omega}{\omega_m'}$ and $\chi_m = \frac{\omega^*}{\omega_m'} = \frac{\nu_2}{k\omega_m'}$, we rewrite equality (36) in the form

$$\lambda_{mn} = \frac{1}{\sqrt{(1 - z_m^2)^2 + z_m^2 \chi_m^2}}.$$

The quantity λ_m reaches a maximum at $z_m = \sqrt{1 - \frac{1}{2}\chi_m^2}$. Thus, resonance frequencies are determined by the formulas

$$\omega_{m\mathrm{r}} = \omega_m' \sqrt{1 - \frac{1}{2}\left(\frac{\nu_2}{k\omega_m'}\right)^2}, \quad m = 1, 2, 3, \ldots . \tag{37}$$

In this case, the maximum values of the dynamic-response factor are

$$\lambda_{m\,\max} = \frac{\omega'_m}{\omega^*}\left(1 - \frac{1}{2}\frac{\omega^{*2}}{\omega'^2_m}\right)^{-\frac{1}{2}} .$$

Resonance frequencies at which the coefficients λ_{mn} reach a maximum are also found from expression (37).

As is seen from formula (37), resonance frequencies of vibrations of the air filling the pores in the studied snow specimen do not coincide with any of the natural frequencies of vibrations of air $\omega'_m = \pi m c_2/h$ in the absence of the solid skeleton of snow. Moreover, it follows from this formula that in the specified specimen of snow resonance vibrations of air inclusions of snow can be excited only when the radicand in (37) is positive, i.e., if $\frac{\nu_2}{k\omega'_m} < \sqrt{2}$. Hence, we obtain

$$\frac{k}{h} > \frac{\nu_2}{\sqrt{2}\,\pi m c_2}\,, \quad m = 1, 2, 3, \ldots \,. \tag{38}$$

If condition (38) is satisfied, then condition (33) is also satisfied automatically. Thus, for resonance to appear in the air confined in the snow specimen of thickness h, condition (33), which is the necessary condition for the appearance of damping vibrations in the snow specimen, is inadequate. For resonance to originate, the ratio k/h must satisfy a more strict requirement (38). In order for the first resonance to appear, the condition $\frac{h}{k} < \frac{\sqrt{2}\,\pi c_2}{\nu_2}$ must hold, for the second resonance — $\frac{h}{k} < \frac{2\sqrt{2}\,\pi c_2}{\nu_2}$, and so on. Substituting the values $\nu_2 = 2\cdot 10^{-5}$ m^2/sec and $c_2 = 320$ m/sec in these conditions, we obtain $h < 7.2\cdot 10^7 k$ and $h < 14.4\cdot 10^7 k$ (k, m^2; h, m), respectively. These conditions have a quite definite physical meaning.

Origination of the steady forced vibrations of the air mass filling the pores in the snow specimen requires the snow density to be rather small. At large snow densities, elastic vibrations that are excited on the lower base of the studied snow specimen damp at a distance of the order of the thickness h of the snow specimen, not reaching its surface; under these conditions, the concept of resonance loses its meaning. As the snow density increases, k decreases sharply.

In the experimental studies [1], several resonance frequencies related to elastic vibrations of the air filling the snow pores were clearly recorded for some specimens of snow. In individual snow specimens with rather large densities, resonance frequencies were not recorded. From the point of view of the theory formulated above, these results are quite explicable.

The main advantage of the technique suggested in [1] is that it can be used for determining the coefficient of permeability of snow k, which is expressed in terms of the resonance frequency ν_{1r}:

$$k = \frac{\dfrac{\nu_2 h}{\sqrt{2}\,\pi c_2}}{\sqrt{1 - \left(\dfrac{2h\nu_{1r}}{c_2}\right)}}\,. \tag{39}$$

Hence it follows that the dimensionless combination of the parameters $\chi = \chi_{1r} = \frac{\nu_2}{k\omega_{1r}}$, which corresponds to the resonance frequency ω_{1r}, is

$$\chi_{1r} = \sqrt{2}\sqrt{\left(\frac{c_2}{2h\nu_{1r}}\right)^2 - 1} \ . \tag{40}$$

On the other hand, the dependence of the dimensionless pore velocity of sound in snow v_g/c_2 on the parameter χ is presented in the form of a universal curve, which is given in [2]. Using this curve, one can find the ratio v_g/c_2, which corresponds to the value χ_{1r}, and consequently, the pore velocity of sound v_g in the studied snow specimen.

Thus, formulas (39) and (40) allow one, using the technique suggested in [1], to determine the coefficient of permeability of the studied snow specimen and the velocity of sound v_g in the air confined in the snow pores. In this case, one should bear in mind that as the snow density increases, the coefficient of permeability k decreases, and consequently the required thickness of the studied snow specimen decreases in the same proportion. This will result in an increase in the operating range of frequencies in the experimental setup [1].

NOTATION

A, dimensionless quantity; a and b, arbitrary constants; c_2, velocity of sound in air; c_l, longitudinal velocity of sound in snow, m/sec; E, Young's modulus, Pa; F_0, force amplitude, N/m^2; f_0, coefficient of snow porosity; h, thickness of the snow specimen, m; k, coefficient of permeability of snow, m^2; t, time, sec; u_1 and u_2, small vertical displacements of the elements of the structures of the snow skeleton and air inclusions in snow, m; v_g, velocity of sound in the air confined in the snow pores (pore velocity of sound), m/sec; x, coordinate, m; δ_m, shift of a phase (m = 1, 2, 3, ...); ε and ε', dimensionless small quantities of order 10^{-2}–10^{-3}; λ_{mn}, dynamic-response factors (m, n = 1, 2, 3, ...); $\lambda_{m\,\max}$, maximum value of the parameter λ_m (m = 1, 2, 3, ...); ν_2, kinematic coefficient of viscosity, m^2/sec; ν_{1r}, first resonance frequency of vibrations of the solid skeleton of the snow specimen, sec^{-1}; ν_{nr}, nth resonance frequency of vibrations of the solid skeleton of the snow specimen (n = 1, 2, 3, ...), sec^{-1}; ξ, integration variable, m; ρ, snow density, kg/m^3; σ, Poisson coefficient; τ, variable of integration with respect to time, sec; ω, frequency of the outer force, sec^{-1}; $\omega^* = \nu/k$, quantity having the dimension of sec^{-1}; ω_n, natural frequencies of vibrations of the solid skeleton of the studied snow specimen (n = 1, 2, 3, ...), sec^{-1}; ω_{1r}, first angular resonance frequency of vibrations of the solid skeleton of the specimen, sec^{-1}; ω_{nr}, nth angular resonance frequency of vibrations of the solid skeleton of the studied snow specimen (n = 1, 2, 3, ...), sec^{-1}; $\bar{\omega}_m = \sqrt{\omega_m'^2 - \frac{1}{4}\omega^{*2}}$ (m = 1, 2, 3, ...), sec^{-1}; ω_m', natural frequencies of vibrations of the air layer of thickness h (m = 1, 2, 3, ...), sec^{-1}; ω_{mr}, resonance frequencies of vibrations of the air confined in the snow pores (m = 1, 2, 3, ...), sec^{-1}. Indices: l, longitudinal; g, group; max, maximum; r, resonance; m = 1, 2, 3, ..., refers to air inclusions in snow; n = 1, 2, 3, ..., refers to the solid skeleton.

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